stochastic processes and their



# Nonlinear self-stabilizing processes – I Existence, invariant probability, propagation of chaos

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Received 23 January 1997; accepted 13 January 1998

**Abstract**

Taking an odd, non-decreasing function $\beta$, we consider the (nonlinear) stochastic differential equation

$$\begin{aligned} &X_t = X_0 + B_t - \frac{1}{2}\int_0^t \beta * u(s, X_s)\mathrm{d}s, \quad t \geqslant 0, \\ &P(X_t \in \mathrm{d}x) = u(t, \mathrm{d}x), \quad t > 0, \end{aligned} \qquad (\widetilde{\mathrm{E}})$$

and we prove the existence and uniqueness of solution of Eq. $(\widetilde{\mathrm{E}})$, where $\beta * u(s,x) = \int_{\mathbb{R}} \beta(x-y)u(s, \mathrm{d}y)$ and $(B_t;\ t \geqslant 0)$ is a one-dimensional Brownian motion, $B_0 = 0$. We show that Eq. $(\widetilde{\mathrm{E}})$ admits a stationary probability measure and investigate the link between Eq. $(\widetilde{\mathrm{E}})$ and the associated system of particles. © 1998 Published by Elsevier Science B.V. All rights reserved.

*AMS classification*: 60H10; 60K35; 60J60

## 1. Introduction

(1) Let $\beta : \mathbb{R} \to \mathbb{R}$ be an odd non-decreasing function. $(B_t,\ t \geqslant 0)$ will denote a standard Brownian motion, $B_0 = 0$. We are interested here in the following system:

$$\begin{aligned} &X_t = X_0 + B_t - \frac{1}{2}\int_0^t \beta * u(s, X_s)\,\mathrm{d}s, \quad t \geqslant 0, \\ &P(X_t \in \mathrm{d}x) = u(t,\ \mathrm{d}x). \end{aligned} \qquad (\widetilde{\mathrm{E}})$$

where $\beta * u$ denotes the convolution between $\beta$ and $u$:

$$\beta * u(s,x) = \int_{\mathbb{R}} \beta(x-y)u(s, \mathrm{d}y).$$

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0304-4149/98/$19.00 
PII: S0304-4149(98)00018-0

In Eq. ($\widetilde{\mathrm{E}}$) there are two unknown parameters: the process $(X_t;\ t\geqslant 0)$ and the family of measures $(u(t,.);\ t\geqslant 0)$. Using the Itô formula, it is easy to show that $u$ satisfies the non-linear equation

$$\begin{aligned} &u_t = \tfrac{1}{2}u_{xx} + \tfrac{1}{2}[(\beta * u)u]_x,\\ &u(0,\mathrm{d}x) = P(X_0 \in \mathrm{d}x). \end{aligned} \tag{F}$$

The system ($\widetilde{\mathrm{E}}$) can be reduced to the following stochastic differential equation (SDE):

$$\begin{aligned} &X_t = X_0 + B_t - \frac{1}{2}\int_0^t b(s,X_s)\,\mathrm{d}s,\\ &b(s,x) = E[\beta(x - X_s)]. \end{aligned} \tag{E}$$

From now on we only deal with Eq. (E) instead of Eq. (F).

Similar processes have been considered in the following contexts:

(a) $\beta$ is a bounded Lipschitz continuous function. Eq. (F) is a Mc Kean Vlasov equation (Mc Kean, 1966).

(b) $\beta$ is the Dirac measure at 0. Eq. (F) is the Burgers equation, modelling locally interacting particles (Stroock and Varadhan, 1979).

(c) $\beta$ is the derivative of the Dirac measure at 0. Eq. (F) is the Ölschläger equation remodelling locally interacting particles (Orelschläger, 1985).

(d) Funaki (1984) proved existence and uniqueness of weak solutions of Eq. (E), in $\mathbb{R}^d$. However, the assumptions of Funaki are different from ours.

One-dimensional stationary Mc Kean–Vlasov-type equations (1966), are of the type

$$v_{xx} + ((\beta * v)v)_x = 0. \tag{1.1}$$

Two natural questions are as follows.

(a) Consider the integro-differential equation

$$\frac{\partial u}{\partial t} = v_{xx} + ((\beta * v)v)_x. \tag{1.2}$$

Under which conditions on the kernel $\beta$ do we have

$$w - \lim_{t\to\infty} u(t,x) = v(x). \tag{1.3}$$

(b) Besides, consider the interacting particles system

$$X_t^{i,N} = X_0^i + B_t^i - \frac{1}{2}\int_0^t \left\{ \frac{1}{N}\left( \sum_{j=1}^N \beta(X_s^{i,N} - X_s^{j,N}) \right) \right\} \mathrm{d}s.$$

If the propagation of chaos holds, then the sequence of empirical measures $((1/N)\sum_{i=1}^N \delta_{X_t^{i,N}})$, on the space $C([0,T];\mathbb{R})$ ($T$ fixed), converges in law, as $N$ goes to infinity, to a deterministic probability measure $\mu$, and moreover one has that $\mu_t(\mathrm{d}x) = u(t,x)\,\mathrm{d}x$, $\forall t\in[0,T]$. Thus, if Eq. (1.3) holds, and if the approximation error of $u(t,x)\,\mathrm{d}x$ by $(1/N)(\sum_{i=1}^N \delta_{X_t^{i,N}})$ can be controlled (in some reasonable sense) uniformly with respect to $t\in[0,+\infty[$, then a probabilistic numerical procedure to solve Eq. (1.1) could consist in simulating the particles system $(X_t^{i,N})$ for $N$ and $t$ large enough. In the present paper, we only address the question (a).

(2) In this paper, we assume

$$\beta \text{ is an odd, increasing, locally Lipschitz continuous function with polynomial growth.} \tag{1.4}$$

Tamura (1984, 1987) (cf. also Dawson, 1983) analyzed an equation resembling Eq. (E). However this author considered an Ornstein–Ulhenbeck process (converging at infinity) altered by a bounded non-linear drift term. Our context is different: we disturb a non-convergent process – the Brownian motion – by an unbounded non-linear disturbance.

We claim that the two assumptions I and II of Funaki (1984) are not satisfied in our context. Let $u$ and $v$ be two even functions. If we choose $\beta(x)=x^3$, for instance, it is not difficult to check that (2.1) of assumption I is not verified. If we take $\beta(x)=x^5$, then $x \to b(x,u)$ grows as $x^3$, therefore assumption II(ii) of Funaki (1984) does not hold.

(3) Section 2 is devoted to the easiest case: $\beta(x)=ax+b$. For this choice of $\beta$, we can calculate $X$ and $u$ explicitly, and it is obvious that $X_t$ converges if and only if $a>0$ and $b=0$. Moreover, the limit is a Gaussian distribution. This example shows that $X_t$ converges if

$$\beta(0)=0; \quad \operatorname{sgn}(x)\beta(x)\geqslant 0 \quad \forall x\in\mathbb{R}.$$

We prove in Section 3 the existence and uniqueness of Eq. (E) when $\beta$ satisfies moreover

$$\beta(x)-\beta(y)\geqslant\beta_1(x-y)+\beta_0 \quad \forall x\geqslant y \tag{1.5}$$

where $\beta_1>0$ and $\beta_0\in\mathbb{R}$.

Let $(X,b)$ be a solution of Eq. (E). Assume for simplicity that the distribution of $X_0$ is symmetric (i.e. $X_0$ and $-X_0$ have the same law). Then $b(t,.)$ is an odd function and

$$\operatorname{sgn}(x)\; b(t,x)\geqslant 0 \quad \forall x\in\mathbb{R},\ \forall t\geqslant 0. \tag{1.6}$$

This property plays a crucial role, and replaces the lack of bounds for $\beta$.

In the general case, we associate $b$ with the solution $X^{(b)}$ of the (classical) stochastic differential equation

$$X_t^{(b)}=X_0+B_t-\frac{1}{2}\int_0^t b(s,X_s^{(b)})\,\mathrm{d}s. \tag{1.7}$$

We set $\Gamma b(s,x)=E[\beta(x-X_s^{(b)})]$. We check that $\Gamma$ has a fixed point, this allows us to prove that Eq. (E) admits a unique solution.

In Section 4, we investigate the existence of an invariant measure. The existence requires the convexity of $\beta$ only (we make use of a fixed point theorem based on Schauder theorem). To prove the uniqueness, we assume in addition that $\beta(x)=\beta_0(x)+\alpha x$, where $\beta_0$ is an odd, increasing function, Lipschitz continuous, with linear growth, and $\alpha$ is positive and large enough. This hypothesis is not necessary, since there exists a unique invariant probability measure in the two special cases: $\beta_0(x)=x^3$ and $\beta_0(x)=x^5$.

In the last section we associate a system of particles with Eq. (E) and we prove that the limit system has the propagation chaos property (we do not need to suppose that Eq. (5) holds).

In this context, processes $X$ solving Eq. (E) cannot be used to model the position of interacting particles. If two particles are in $x$ and $y$, the drift equals $\beta(x-y)$, and is an increasing function of $x-y$, which is not very physical. However, $X$ can represent the charge of ionized particles lying in a chemical or biological medium. Suppose that two particles have charge $x$ being greater than $y$. These particles interact, i.e. electrons come from particle 2 to particle 1, therefore the charge of particle 1 (resp. 2) decreases (resp. increases). Moreover, the flux of electrons is stronger if the difference of charges is greater. It seems intuitive that this system tends toward have an equilibrium state. This limit state is not given a priori, and depends only on the exchange of charges (i.e. function $\beta$) and the initial data. More precisely, we prove that the equilibrium state depends only on $\beta$ and $E(X_0)$.

In a second paper, we investigate the convergence of $X_t$, in distribution, when $t$ goes to infinity, to the stationary probability.

## 2. The case $\beta(x) = ax + b$

We observe that $b(s,x)=E(\beta(x-X_s))=ax+b-am(s)$, where $m(s)=E(X_s)$. As a result (E) is now equivalent to

$$X_t = X_0 + B_t - \frac{a}{2}\int_0^t (X_s - m(s))\,\mathrm{d}s - \frac{bt}{2}.$$

We take the expectation of both sides

$$m(t) = E(X_t) = m(0) - \frac{bt}{2}.$$

This means that $m$ is known. We set

$$Y_t = X_t - m(0) + \frac{bt}{2}. \tag{2.1}$$

$Y$ satisfies the following linear stochastic differential equation:

$$Y_t = Y_0 + B_t - \frac{a}{2}\int_0^t Y_s\,\mathrm{d}s.$$

The explicit solution is given by

$$Y_t = \mathrm{e}^{-at/2}Y_0 + Z_t; \quad Z_t = \mathrm{e}^{-at/2}\int_0^t \mathrm{e}^{as/2}\,\mathrm{d}B_s.$$

$Z_t$ is a Gaussian r.v. with mean 0 and variance $\sigma(t)^2 = (1-\mathrm{e}^{-at})/a$. It is now clear that if $a<0$, $Y_t$ does not converge, in distribution, $t\to\infty$. We assume that $a>0$. Hence $Z_t$ converges to $\mathcal{N}(0,1/a)$. Consequently, $X_t$ converges if and only if $b=0$. Finally if $a>0$ and $b=0$, $X_t$ converges, in distribution, to the Gaussian distribution $\mathcal{N}(m(0),1/a)$, as $t$ approaches infinity.

It is interesting to note that the limit distribution depends on $X_0$ through $m(0) = E(X_0)$. If $E(X_0) = 0$, we have

$$E[(X_t - \mathrm{e}^{-at/2} X_0)^{2n}] \leqslant \left(\frac{1 - \mathrm{e}^{-at}}{a}\right)^n E((B_1)^{2n}).$$

Consequently,

$$\sup_{t \geqslant 0} E[|X_t|^{2n}] \leqslant c_n (1 + E(|X_0|^{2n})). \tag{2.2}$$

## 3. Existence and uniqueness of solutions of Eq. (E)

In this section we assume that $\beta : \mathbb{R} \to \mathbb{R}$ satisfies:

$$\beta \quad \text{is increasing,} \tag{3.1}$$

$$\beta \quad \text{is an odd function} \quad (\beta(x) = -\beta(-x)\ \forall x \in \mathbb{R}). \tag{3.2}$$

There exist $c > 0$, $r \in \mathbb{N}^*$, $\beta_1 > 0$, $\beta_0 \in \mathbb{R}$ such that

$$|\beta(x) - \beta(y)| \leqslant |x - y|(c + |x|^r + |y|^r) \quad \forall x, y \in \mathbb{R}, \tag{3.3}$$

$$\beta(x) - \beta(y) \geqslant \beta_1 (x - y) + \beta_0 \quad \forall x \geqslant y. \tag{3.4}$$

If $\beta$ is a polynomial function, then Eq. (3.3) holds. If $\beta$ is an increasing and $C^1$ function such that $\beta'(x) \geqslant \beta_1 > 0$, for $|x|$ large enough, then Eq. (3.4) is satisfied. We now state the main result of this section:

**Theorem 3.1.** *Let $X_0$ be a r.v. such that $E[X_0^{2(r+1)^2}] < \infty$. Then the non-linear SDE*:

$$\begin{aligned} X_t &= X_0 + B_t - \frac{1}{2}\int_0^t b(s, X_s)\,\mathrm{d}s, \\ b(s,x) &= E[\beta(x - X_s)] \end{aligned} \tag{E}$$

*has a unique strong solution.*

**Remarks 3.2.** (a) If $\beta$ is a bounded and Lipschitz continuous function, Sznitzman and Varadhan (1986) proved this theorem using the Vasertein metric. In our context, we "replace" the boundedness of the coefficient by the fact that $x\beta(x)$ is positive and large as $x$ goes to $\pm\infty$.

(b) We observe that if $E(|X_t|) < \infty$, then

$$E(X_t) = E(X_0) \quad \forall t \geqslant 0, \text{ and } X_t - E(X_0) \text{ verifies Eq. (E).}$$

To prove this, we take the expectation in Eq. (E):

$$E(X_t) = E(X_0) - \frac{1}{2}\int_0^t E(\beta(X_s - X_s'))\,\mathrm{d}s,$$

where $X_s'$ is an independent copy of $X_s$. $\beta$ being an odd function, $E(\beta(X_s'-X_s))=0$. Therefore $E(X_t)=E(X_0)$. It is easy to check that $X_t-E(X_0)$ solves Eq. (E). As a result we can suppose that $E(X_t)=E(X_0)=0$, $\forall t\geqslant 0$.

The first step in our proof of Theorem 3.1 consists in checking the uniqueness and existence on an interval $[0,T]$, $T$ being "small". Then we extend these properties to $\mathbb{R}_+$. In the following, we will need the classical result (see, for instance, Stroock and Varadhan, 1979, Theorem 10.2.2, p. 255):

**Proposition 3.3.** *Let $b:\mathbb{R}_+\times\mathbb{R}\to\mathbb{R}$ be a function such that for ever $n$*

$$\max_{s\geqslant 0}|b(s,0)|<\infty, \tag{3.5}$$

$$|b(s,x)-b(s,y)|\leqslant c_n|x-y|, \tag{3.6}$$

*for every $n$, $|x|\leqslant n$, $|y|\leqslant n$ and*

$$\operatorname{sgn}(x)\ b(s,x)\geqslant 0 \quad \textit{for } |x| \textit{ large enough.} \tag{3.7}$$

*Then the (classical) SDE:*

$$X_t^{(b)}=X_0+B_t-\frac{1}{2}\int_0^t b(s,X_s^{(b)})\,\mathrm{d}s \tag{F$_\mathrm{b}$}$$

*admits a unique strong solution, for any initial data $X_0$.*

The idea of the proof of Theorem 3.1 is the following: we associate with a function $b$, the solution $X^{(b)}$ of Eq. (F$_\mathrm{b}$) and set

$$\Gamma b(s,x)=E[\beta(x-X_s^{(b)})].$$

We prove that $\Gamma$ has a fixed point $b$. Consequently $X^{(b)}$ is a solution of Eq. (E).

More precisely, we have to control the moments of $X_t^{(b)}$ and to define an appropriate functional space $\Lambda_T$ such that the restriction of $\Gamma$ to $\Lambda_T$ is a contraction. The proof of Theorem 3.1 is divided in five steps: Lemmas 3.4–3.8.

**Notation.**
1. If $\beta$ satisfies Eq. (3.3), let $2q\geqslant r+1$, then there exists $C>0$ such that

$$|\beta(x)|\leqslant C(1+|x|^{2q}) \quad \forall x\in\mathbb{R}. \tag{3.8}$$

2. For any positive $T$, and $b:[0,T]\times\mathbb{R}\to\mathbb{R}$ we set

$$\|b\|_T=\sup_{0\leqslant s\leqslant T}\sup_{x\in\mathbb{R}}\left(\frac{|b(s,x)|}{1+|x|^{2q}}\right), \quad 2q\geqslant r+1. \tag{3.9}$$

3. Let $\wedge_T$ be the set of $b:[0,T]\times\mathbb{R}\to\mathbb{R}$ such that

$$x\to b(s,x) \text{ is a non-decreasing function, for every } s\in[0,T], \tag{3.10}$$

- $b(s,.)$ is a locally Lipschitz continuous function, uniformly with respect to $s\in[0,T]$:

$$|b(s,x)-b(s,y)|\leqslant c_n|x-y|,\quad \forall s\in[0,T],\quad |x|\leqslant n,\ |y|\leqslant n, \tag{3.11}$$

$$b(s,x)-b(s,y)\geqslant \beta_1(x-y)+\beta_0\quad \forall s\in[0,T],\quad \forall x\geqslant y, \tag{3.12}$$

$$\|b\|_T<\infty. \tag{3.13}$$

4. If $b$ satsifies Eqs. (2.5)–(2.7), and $X^{(b)}$ is the solution of Eq. ($\mathrm{F_b}$), we introduce

$$v_n^b(t)=E[|X_t^{(b)}|^n], \tag{3.14}$$

$$\hat{v}_n^b(t)=\sup_{0\leqslant s\leqslant t} v_n^b(s). \tag{3.15}$$

5. From now on, $k_1,k_2,k_3,\ldots$ are "universal" constants, this means that $k_n$ depends only on the fixed function $\beta$. In the same way $k_1(.)$ denotes a universal function.

**Lemma 3.4.** *Assume that* $b\in\wedge_T$, $n\geqslant 1$, $\rho(x)=\beta_0 x$. *Then $b$ satisfies Eq.* (3.5)–(3.7), $\rho\in\wedge_T$, $\hat{v}_{2n}^{\rho}(T)<\infty$ *and*

$$\hat{v}_{2n}^b(T)\leqslant k_1(n)(\hat{v}_{2n}^{\rho}(T)+(T\|b-\rho\|_T)^{2n}(1+\hat{v}_{4qn}^{\rho}(T))),$$

**Proof of Lemma 3.4.**
1. Let $b$ be an element of $\wedge_T$, then $b$ verifies $|b(s,0)|\leqslant\|b\|_T$ and

$$\operatorname{sgn}(x)\ b(s,x)\geqslant\beta_1|x|+\beta_0-\|b\|_T.$$

Consequently Eqs. (2.5)–(2.7) holds.

2. Suppose $b\in\wedge_T$, $f\in\wedge_T$. $X^{(b)}$ and $X^{(f)}$ being solutions of Eq. ($\mathrm{F_b}$) respectively ($F_f$), then

$$X_t^{(b)}-X_t^{(f)}=-\frac{1}{2}\int_0^t (b(s,X_s^{(b)})-f(s,X_s^{(f)}))\,\mathrm{d}s.$$

For every $\alpha>1$, $x\to|x|^\alpha$ is a $C^1$-function, hence $|X_t^{(b)}-X_t^{(f)}|^\alpha$ is equal to

$$-\frac{\alpha}{2}\int_0^t \operatorname{sgn}(X_s^{(b)}-X_s^{(f)})|X_s^{(b)}-X_s^{(f)}|^{\alpha-1}1_{\{X_s^{(b)}\neq X_s^{(f)}\}}(b(s,X_s^{(b)})-f(s,X_s^{(f)}))\,\mathrm{d}s.$$

We take the limit, $\alpha\to 1+$,

$$\begin{aligned}&|X_t^{(b)}-X_t^{(f)}|\\ &\quad=-\frac{1}{2}\int_0^t \operatorname{sgn}\ (X_s^{(b)}-X_s^{(f)})(b(s,X_s^{(b)})-b(s,X_s^{(f)})\\ &\qquad+b(s,X_s^{(f)})-f(s,X_s^{(f)}))\,\mathrm{d}s.\end{aligned}$$

Since $x\to b(s,x)$ is non-decreasing, sgn $(x-y)(b(s,x)-b(s,y))\geqslant 0$. Then

$$|X_t^{(b)}-X_t^{(f)}|\leqslant\frac{1}{2}\int_0^t |b(s,X_s^{(f)})-f(s,X_s^{(f)})|\,\mathrm{d}s.$$

Eq. (3.9) implies

$$|X_t^{(b)} - X_t^{(f)}| \leqslant \frac{1}{2}\|b-f\|_T \int_0^t (1+|X_s^{(f)}|^{2q}))\,\mathrm{d}s. \tag{3.16}$$

Moreover, using Hölder inequalities, we have

$$E\left[\left(\int_0^t (1+|X_s^{(f)}|^{2q})\,\mathrm{d}s\right)^{2n}\right] \leqslant k_2(n)T^{2n}(1+\hat{v}_{4qn}^{f}(T)), \tag{3.17}$$

for any $0\leqslant t\leqslant T$.

3. We choose $f(s,x)=\rho(x)=\beta_0 x$. Then $f\in\wedge_T$, and Eq. (1.2) tells us that $\hat{v}_{2n}^{\rho}(T)<\infty$. Using a convexity argument we easily obtain from Eqs. (3.16) and (3.17),

$$E[(X_t^{(b)})^{2n}] \leqslant \hat{v}_{2n}^{\rho}(T) + k_3(n)T^{2n}(1+\hat{v}_{4qn}^{\rho}(T))\|b-\rho\|_T^{2n}. \qquad \square$$

If $b\in\wedge_T$, we set

$$\Gamma b(s,x) = E[\beta(x - X_s^{(b)})]. \tag{3.18}$$

The function $\Gamma b$ is well defined since Eq. (3.8) holds.

**Lemma 3.5.**

1. $\Gamma$ *maps* $\wedge_T$ in $\wedge_T$ (i.e. $\Gamma(\wedge_T)\subset\wedge_T$) *and*

$$\|\Gamma b\|_T \leqslant k_4(1+\hat{v}_{2q}^{b}(T)). \tag{3.19}$$

2. $\Gamma$ is Lipschitz continuous: there exists $k_5:\mathbb{R}_+^2\to\mathbb{R}_+$, $k_5(x,.)$, $k_5(.,x)$ being increasing functions for every $x\geqslant 0$, such that $\forall b\in\wedge_T$, $\forall f\in\wedge_T$,

$$\|\Gamma b - \Gamma f\|_T \leqslant \|b-f\|_T T\, k_5(\hat{v}_{4q}^{b}(T), \hat{v}_{4q}^{f}(T)). \tag{3.20}$$

**Proof of Lemma 3.5.**

1. We set $\Gamma b = c$. Lemma 3.4 and (3.8) (resp. (3.3)) imply $\|c\|_T<\infty$ (resp. (3.11) is satisfied). Since $\beta$ is increasing, $c(s,.)=E[\beta(.-X_s^{(b)})]$ is also non-decreasing. Suppose that $x\geqslant y$, since $\beta$ satisfies Eq. (3.4), and that $x-X_s^{(b)}\geqslant y-X_s^{(b)}$. Then

$$c(s,x)-c(s,y) = E[\beta(x-X_s^{(b)}) - \beta(y-X_s^{(b)})] \geqslant \beta_1(x-y)+\beta_0.$$

We have verified $\Gamma b\in\wedge_T$. By Eq. (3.8), we have

$$|\Gamma b(s,x)| \leqslant E[|\beta(x-X_s^{(b)})|] \leqslant k_4(1+x^{2q})E(1+|X_s^{(b)}|^{2q}).$$

Consequently, Eq. (3.19) is verified.

2. Let $b,f$ in $\wedge_T$. We set $X=X^{(b)}$ and $Y=X^{(f)}$ for simplicity. Using Eq. (3.3), we have

$$\begin{aligned}|\Gamma b(s,x) - \Gamma f(s,x)| &\leqslant E[|\beta(x-X_s) - \beta(x-Y_s)|]\\ &\leqslant k_6 E[|X_s-Y_s|(1+|x|^r+|X_s|^r+|Y_s|^r)]\\ &\leqslant k_7(1+|x|^r)E(|X_s-Y_s|(1+|X_s|^r+|Y_s|^r))\\ &\leqslant k_8(1+|x|^r)\{E(|X_s-Y_s|^2)(1+E(|X_s|^{2r})+E((Y_s)^{2r}))\}^{1/2}.\end{aligned}$$

Using Lemma 3.4, we obtain

$$\|\Gamma b - \Gamma f\|_T \leqslant k_8(1 + \hat{v}^b_{2r}(T) + \hat{v}^f_{2r}(T))^{1/2}(E(|X_s - Y_s|^2))^{1/2}.$$

Previous estimates (3.16) and (3.17) (with $n=1$) both yield

$$E((X_s - Y_s)^2) \leqslant k_9 \|b - f\|^2_T T^2 (1 + \hat{v}^f_{4q}(T)).$$

Moreover,

$$\hat{v}^b_{2r}(T) \leqslant (\hat{v}^c_{4q}(T))^{r/2q}, \quad c = f \text{ or } b. \qquad \square$$

**Lemma 3.6.** *Let* $K \geqslant 2k_4(1+k_1(q))\hat{v}^\rho_{2q}(\infty)$ and $\wedge^K_T = \wedge_T \cap \{b; \|b\|_T \leqslant K\}$. *There exists* $k_{10}$ *such that if* $T = k_{10}(K; E(|X_0|^i), 1 \leqslant i \leqslant 8q^2)$. *Then*

(i) $\Gamma(\wedge^K_T) \subset \wedge^K_T$ *and the Lipschitz norm of* $\Gamma$ *restricted to* $\wedge^K_T$ *is less than* $\frac{1}{2}$.

(ii) *There exists a strong solution of* Eq. (E) *such that*

$$\sup_{0 \leqslant t \leqslant T} E(|X_t|^{2q}) < \infty. \tag{3.21}$$

**Proof of Lemma 3.6.**

1. Using Eq. (3.19) and Lemma (3.4) successively, we have

$$\|\Gamma b\|_T \leqslant k_4\{1 + k_1(q)[\hat{v}^\rho_{2q}(\infty) + 2^{2q}T^{2q}(\|b\|^{2q}_T + \|\rho\|^{2q}_\infty)(1 + \hat{v}^\rho_{4q^2}(\infty))]\},$$

where $\|\rho\|_\infty = \sup_{x>0} |\beta_0 x|/(1 + |x|^{2q})$, and

$$\hat{v}^\rho_n(\infty) = \sup_{t \geqslant 0} E(|X^\rho_t|^n).$$

If we choose $T$ small enough such that

$$k_4 k_1(q) 2^{2q}(K^{2q} + \|\rho\|^{2q}_\infty)(1 + \hat{v}^\rho_{4q^2}(\infty))T^{2q} \leqslant K/2 \tag{3.22}$$

then $\|\Gamma b\|_T \leqslant K$, $b$ being an element in $\wedge^K_T$. We have proved that $\Gamma(\wedge^K_T) \subset \wedge^K_T$.

2. Let $k_{11}$ be the function defined by $k_{11}(x) = k_5(x,x)$ ($k_5$ appears in Eq. (3.20)). Eq. (3.20) and Lemma 3.4 both imply $|||\Gamma|||_T \leqslant \frac{1}{2}$ if $T$ satisfies

$$T k_{11}(k_1(2q)\{\hat{v}^\rho_{4q}(\infty) + 2^{4q}(K^{4q} + \|\rho\|^{4q}_\infty)(1 + \hat{v}^\rho_{8q^2}(\infty))T^{4q}\}) \leqslant \tfrac{1}{2}. \tag{3.23}$$

Since $\rho(x) = \beta_0 x$, (i) is a consequence of the above inequalities.

3. We assume that $T = k_{10}(K; E(|X_0|^k), 1 \leqslant k \leqslant 8q^2)$. We will now establish that Eq. (E) admits a strong solution. Let $b_0$ be an element of $\wedge^K_T$. By induction, we define the sequence $(b_n)$:

$$b_{n+1} = \Gamma b_n.$$

Since the Lipschitz norm of $\Gamma$ is less than $\frac{1}{2}$, $(b_n)$ is a Cauchy sequence belonging to $\wedge^K_T$. It converges, with respect to $\|.\|_T$ to $b$, verifying Eqs. (3.10), (3.12) and

$\|b\|_T \leqslant K$. We claim $b \in \wedge_T^K$: we have to show that $b$ satisfies the Lipschitz property Eq. (3.11). Let $X_n = X^{(b_n)}$. Since $b_{n+1} = \Gamma b_n$,

$$\begin{aligned} |b_{n+1}(s,x) - b_{n+1}(s,y)| &\leqslant E[|\beta(x - X_n(s)) - \beta(y - X_n(s))|] \\ &\leqslant k_{12}(N)(1 + \hat{v}_r^{b_n}(T))|x-y|, \end{aligned}$$

if $|x| \leqslant N$, $|y| \leqslant N$.
Since $\|b_n\|_T \leqslant K$, Lemma 3.4 implies

$$|b_{n+1}(s,x) - b_{n+1}(s,y)| \leqslant k_{13}(N, \hat{v}_{4q^2}^{\rho}(\infty), K, T)|x-y|.$$

We can take the limit as $n \to \infty$, and obtain

$$|b(s,x) - b(s,y)| \leqslant k_{13}(N, \hat{v}_{4q^2}^{\rho}(\infty), K, T)|x-y|.$$

It is obvious that $b = \Gamma b$ and $X = X^{(b)}$ is a strong solution of Eq. (E). □

As part (i) of Lemma 3.6 shows, the constants that appear depend on the moments of $X_0$. Yet we are not able to construct a solution on $[0, +\infty[$. We need to check that these constants do not explode:

$$\sup_{t>0} E(|X_t^{(b)}|^{2n}) < \infty.$$

We start with a preliminary result.

**Lemma 3.7.** *Let $f$ be a continuous and differentiable function defined on* $[0, +\infty[$, *and* $\mathbb{R}$*-valued. We assume that there exists* $l > 0$, *such that* $\{t; f(t) > l\} \subset \{t; f'(t) < 0\}$. *Then*

$$\sup_{x \geqslant 0} f(x) \leqslant f(0) \vee l.$$

**Lemma 3.8.** *Let $b \in \wedge_T$, and suppose $\Gamma(b) = b$ (i.e. $X^{(b)}$ is a solution of* Eq. ($\mathrm{F_b}$) *or* Eq. (E)) *and* $E(X_0) = 0$. *Then*

$$\hat{v}_{2n}^{b}(T) \leqslant k_{14}(m_i;\ 2 \leqslant i \leqslant 2n),$$

*where* $m_k = E(|X_0|^k)$.

**Proof of Lemma 3.8.**

1. Let $X_0$ and $X_0'$ be two independent random variables, having the same distribution. We consider two independent Brownian motions $B$ and $B'$, $X$ and $X'$ which are solutions of

$$X_t = X_0 + B_t - \frac{1}{2}\int_0^t b(s, X_s)\,\mathrm{d}s,$$

and

$$X_t' = X_0' + B_t' - \frac{1}{2}\int_0^t b(s, X_s')\,\mathrm{d}s,$$

respectively, where $b(s,x)=E(\beta(x-X_s))=E(\beta(x-X'_s))$. We set

$$Y_t=X_t-X'_t,\quad \mu_n(t)=E(|Y_t|^n),\quad n\geqslant 2.$$

$Y$ is a semimartingale with decomposition

$$Y_t=Y_0+B_t-B'_t-\frac{1}{2}\int_0^t (b(s,X_s)-b(s,X'_s))\,\mathrm{d}s.$$

We apply the Ito formula and take the expectation and the derivative. We obtain

$$\mu'_{2n}(t)=n\{2(2n-1)\mu_{2n-2}(t)-E[Y_t^{2n-1}(b(t,X_t)-b(t,X'_t))]\}.$$

Suppose that $x\geqslant y$. Since $b$ satisfies Eq. (3.12): $b(t,x)-b(t,y)\geqslant\beta_1(x-y)+\beta_0$, then

$$(x-y)(b(t,x)-b(t,y))\geqslant\beta_1(x-y)^2-|\beta_0||x-y|\quad\forall x\geqslant y.$$

As a result,

$$\mu'_{2n}(t)\leqslant n\{2(2n-1)(\mu_{2n}(t))^{1-1/n}+|\beta_0|(\mu_{2n}(t))^{1-1/2n}-\beta_1\mu_{2n}(t)\}.$$

There exists $k_{15}(n)>0$, such that $x\geqslant k_{15}(n)$ implies

$$2(2n-1)x^{1-1/n}+|\beta_0|x^{1-1/2n}-\beta_1 x<0.$$

Consequently, $\{t;\mu_{2n}(t)>k_{15}(n)\}\subset\{t;\mu'_{2n}(t)<0\}$. Applying Lemma 3.7, we have

$$E[(X_t-X'_t)^{2n}]\leqslant k_{15}(n)\vee E((X_0-X'_0)^{2n}),\quad n\geqslant 1. \tag{3.24}$$

2. Let $\xi$ and $\xi'$ be two independent r.v., $\xi'$ being a copy of $\xi$, such that $E(\xi)=E(\xi')=0$. We claim that

$$E(\xi^{2n})\leqslant k_{16}(E((\xi-\xi')^2),\dots,E((\xi-\xi')^{2n})). \tag{3.25}$$

We will prove this identity by induction on $n$. If $n=1$, $E((\xi-\xi')^2)=2E(\xi^2)$. Then Eq. (3.25) holds. Assume that Eq. (3.25) is satisfied. Since $E(\xi)=E(\xi')=0$, we have

$$E((\xi-\xi')^{2n+2})=2E(\xi^{2n+2})+\sum_{k=2}^{2n}\begin{pmatrix}2n+2\\k\end{pmatrix}E(\xi^k)E(\xi'^{2n+2-k}).$$

This equality implies Eq. (3.25), with $n$ being replaced by $n+1$.

3. We have observed in the Introduction that if $E(X_0)=0$, then $E(X_t)=0$. It is sufficient to use steps 1 and 2 now. □

**Remark 3.9.** Assume $X$ is a solution of $(F_b)$, $X'$ an independent copy of $X$, $b$ being an element of $\Lambda_T$. In general, $E(X_t)\neq E(X_0)$. However, if $\Gamma b=b$ (i.e. $X$ is a solution of Eq. (E)), $E(X_t)=E(X_0)$. This property, as the proof of Lemma 3.8 shows, is crucial to the determination of an upper bound of the moments of $X_t$. If $X$ is a solution of

Eq. (E), the drift term $b(s,X_s)$ is equal to $E(\beta(X_s - X_s'))$. Therefore, it is natural to deal with $X_t - X_t'$.

**Proof of Theorem 3.1.** We can assume $E(X_0)=0$ (see the remark in the Introduction). Let us introduce $U=\max\{T>0;$ Eq. (E) admits a unique solution $X$ on $[0,T]$, $\sup_{0\leqslant t\leqslant T} E(X_t^{2q})<\infty\}$, with the convention $\max\emptyset=0$.

1. We first check $U>0$. We choose $K$:

$$K=\max\{2k_4(1+k_1(q))\hat{v}_{2q}^{\rho}(\infty)),k_4(1+k_{14}(m_i;\ 2\leqslant i\leqslant 2q)))\}. \tag{3.26}$$

By Lemma 3.6, we know there exists $T=k_{17}(m_i;\ 1\leqslant i\leqslant 8q^2)$ and a unique $b\in\Lambda_T^K$ such that $b=\Gamma(b)$. It is clear that $X=X^{(b)}$ is a strong solution of Eq. (E) on $[0,T]$. Assume that $Y$ is a solution of Eq. (E) on $[0,T]$, such that $\sup_{0\leqslant t\leqslant T}E(Y_t^{2q})<\infty$. We set $c(t,x)=E(\beta(x-Y_t)]$. As we check in step 1 of the proof of Lemma 3.5, $c$ belongs to $\Lambda_T$ and

$$|c(t,x)|\leqslant k_4(1+x^{2q})\left(1+\sup_{0\leqslant t\leqslant T}E(Y_t^{2q})\right).$$

Since $c=\Gamma c$, Lemma 3.8 tells us

$$\|c\|_T\leqslant k_4(1+k_{14}(m_i;\ 2\leqslant i\leqslant 2q))\leqslant K.$$

Hence $c\in\Lambda_T^K$, therefore $c=b$ and $Y=X$.

2. We notice that $\hat{v}_{2q}^{\rho}(\infty)=k_{18}(m_i;\ 2\leqslant i\leqslant 2q)$. We set

$$m_i'=k_{14}(m_j;\ 2\leqslant j\leqslant i),\quad i\geqslant 2.$$

Let $K'$ be the positive number defined by Eq. (3.26), where $m_i$ is replaced by $m_i'$. This $K'$ corresponds to $T'=k_{11}(m_i';\ 2\leqslant i\leqslant 8q^2)>0$. Suppose that $U<\infty$. We choose $\varepsilon<T'/2$. There exists $T$, $U-\varepsilon<T<U$, and a unique solution $X$ on $[0,T]$ verifying $\sup_{0\leqslant t\leqslant T}E(X_t^{2q})<\infty$. We consider $(E)$ on $[T,+\infty[$ with initial data $X_T$. By Lemma 3.8,

$$\sup_{0\leqslant t\leqslant T}E(X_t^{2q})\leqslant k_{14}(m_i;\ 2\leqslant i\leqslant 2q).$$

Then as in previous step, we can define a unique solution on $[T,T+T']$. But $T+T'>U$, which generates a contradiction. This shows that $U=\infty$. $\square$

We note that the proof of Theorem 3.1 implies the following result.

**Proposition 3.10.** *Let $X$ be the solution of Eq.* (E). *Suppose that $E(X_0^{2n})<\infty$. Then*

$$\sup_{t\geqslant 0}E(|X_t|^{2n})<\infty\quad\forall n\geqslant 1.$$

## 4. Existence of a stationary distribution

Let $X$ be the solution of *Eq.* (E). Assume that $u(x)\,\mathrm{d}x$ is a stationary distribution. Then $u$ satisfies the Fokker–Plank equation

$$\tfrac{1}{2}u''+\tfrac{1}{2}(u(\beta*u))'=0. \tag{4.1}$$

Integrating this relation, we obtain

$$u(x) = \lambda \exp - \int_0^x (\beta * u)(y)\,\mathrm{d}y. \tag{4.2}$$

If $\beta(x) = x$, then $\beta * u(y) = \int_{\mathbb{R}} (y - z) u(z)\,\mathrm{d}z = y - a$; hence,

$$u(x) = \lambda \exp(-\tfrac{1}{2}x^2 + ax).$$

$u$ being a density function, $\lambda$ is necessarily equal to $\mathrm{e}^{-a^2/2}/\sqrt{2}\pi$.

We observe that we have a family of invariant distributions.

As we do in Section 3, we assume that $\beta$ is a locally Lipschitz continuous, increasing, odd function, and verifies Eq. (3.3).

We note that Eq. (3.2) can be written in the following fore:

$$u(x) = \frac{\exp -(\int_0^x (\beta * u)(y)\,\mathrm{d}y)}{\lambda(u)}, \tag{4.3}$$

where

$$\lambda(u) = \int_{\mathbb{R}} \exp - \left( \int_0^x (\beta * u)(y)\,\mathrm{d}y \right) \mathrm{d}x. \tag{4.4}$$

We set

$$\mathscr{D} = \left\{ v : \mathbb{R} \to \mathbb{R}_+;\ \int_{\mathbb{R}} v(x)\,\mathrm{d}x = 1,\ v(x) = v(-x)\ \forall x \in \mathbb{R},\ \sup_x (1 + |x|^{2n}) v(x) < \infty \right\},$$

with $n$ being large enough, and

$$\mathscr{A}(u)(x) = \frac{\exp -(\int_0^x (\beta * u)(y)\,\mathrm{d}y)}{\lambda(u)}.$$

We remark that if $u = \mathscr{A}(u)$, then $u(x)\,\mathrm{d}x$, is a stationary distribution. We start with an existence result. We have to show that the restriction of $\mathscr{A}$ to a subsequent subset of $\mathscr{D}$ admits a fixed point $u$.

**Theorem 4.1.** *We assume that $\beta$ is a convex function on $[0, +\infty[$ and verifies* Eqs. (3.1)–(3.4).

1. *There exists a symmetric density function $\mu$ (i.e. $\mu(x) = \mu(-x)$, $\forall x \in \mathbb{R}$) satisfying Lemma* 4.3.
2. *If $\mu$ is the density of $X_0$, and $X$ is the unique solution of Eq.* (E), *with initial data $X_0$, then $\mu$ is the density of $X_t$, for any $t \geqslant 0$.*

Our approach is based on Schauder fixed point theorem (Gilburg and Trudinger, 1977, Corollary 11.2, p. 280):

**Proposition 4.2.** *Assume that $\mathscr{B}$ is a Banach space, $\mathscr{C}$ a closed convex subset included in $\mathscr{B}$, $\mathscr{A}$ a map $\mathscr{C} \to \mathscr{C}$ such that*

(i) *$\mathscr{A}$ is continuous,*

(ii) *$\overline{\mathscr{A}(\mathscr{C})}$ is compact. Then $\mathscr{A}$ admits a fixed point in $\mathscr{C}$.*

In order to apply Proposition 4.2, we have to define $(\mathscr{B},\mathscr{C},\mathscr{A})$.

**Notation.**

1. $\mathscr{B}$ is the set of even continuous functions $f:\mathbb{R}\to\mathbb{R}$, such that $\sup_{x\in\mathbb{R}}(1+|x|^p)|f(x)| <\infty$, where $p>4q$ (recall that $2q\geqslant r+1$, $\beta$ satisfies Eqs. (3.3) and (3.8)). $\mathscr{B}$ is equipped with $|.|_\infty : |f|_\infty = \sup_{x\in\mathbb{R}}(1+|x|^p)|f(x)|$.
2. Let $M>0$. We set

$$\mathscr{C}_M = \left\{ f\in\mathscr{B};\ f\geqslant 0,\ f(x)=f(-x),\ \forall x\in\mathbb{R}, \int_{\mathbb{R}} f(x)\,\mathrm{d}x = 1, \right.$$

$$\left. \sup_x (1+|x|^p) f(x) \leqslant M \right\}. \tag{4.5}$$

3. For any $u$ in $\mathscr{C}_M$, we define

$$\gamma_k(u) = \int_{\mathbb{R}} |x|^k u(x)\,\mathrm{d}x, \quad 0\leqslant k\leqslant p-2 \tag{4.6}$$

and

$$\mathscr{A}(u)(x) = \frac{1}{\lambda(u)} \exp - \int_0^x (\beta * u)(y)\,\mathrm{d}y, \tag{4.7}$$

$\lambda(u)$ being defined by Eq. (4.4).

It is clear that $\mathscr{C}_M$ is a closed convex subset of $\mathscr{B}$.

**Lemma 4.3.** *Assume $f$ is an odd function defined on $\mathbb{R}$. Then $f$ is convex on $\mathbb{R}_+$ if and only if*

$$f(x) \leqslant \tfrac{1}{2}(f(x-y)+f(x+y)) \quad \forall x\geqslant 0,\ \forall y\in\mathbb{R}. \tag{4.8}$$

**Lemma 4.4.** *Let $u$ be an element of $\mathscr{C}_M$.*

1. *If* $C_1 = 1+\max_{0\leqslant k\leqslant p-2}\int_{\mathbb{R}}[|x|^k/(1+|x|^p)]\,\mathrm{d}x$, *then*

$$\gamma_k(u) \leqslant MC_1, \quad 0\leqslant k\leqslant p-2. \tag{4.9}$$

2. *$\beta * u$ is an odd function and*

$$\int_0^x \beta(y)\,\mathrm{d}y \leqslant \int_0^x (\beta * u)(y)\,\mathrm{d}y \leqslant C_2 M x^2(1+x^{2q}); \quad \forall x\geqslant 0 \tag{4.10}$$

*where $C_2$ is a constant depending only an $\beta$, and $M$ satisfies*

$$M \geqslant \sup(1, C_1^q). \tag{4.11}$$

**Proof of Lemma 4.4.**

1. It is easy to check Eq. (4.9):

$$\gamma_k(u) = \int_{\mathbb{R}} \frac{|x|^k}{1+|x|^p}(1+|x|^p)u(x)\,\mathrm{d}x \leqslant MC_1, \quad 0\leqslant k\leqslant p-2. \tag{4.12}$$

2. It is clear that $\beta * u$ is an odd function. Let $x \geqslant 0$. We have

$$\beta * u(x) = \beta(x) + \int_{\mathbb{R}} (\beta(x-y) - \beta(x))u(y)\,\mathrm{d}y$$
$$= \beta(x) + \int_0^\infty (\beta(x-y) + \beta(x+y) - 2\beta(x))u(y)\,\mathrm{d}y.$$

$\beta$ being an odd function,

$$\beta * u(x) = \beta(x) + \int_0^\infty (\beta(x+y) - \beta(y-x) - 2\beta(x))u(y)\,\mathrm{d}y. \tag{4.13}$$

This identity will be used later (see the proof of Lemma 4.6).
Recall that $\beta$ is a convex function on $\mathbb{R}_+$, using Lemma 4.3,

$$\beta * u(x) \geqslant \beta(x) \quad \forall x \geqslant 0 \tag{4.14}$$

We immediately deduce the lower bound in Eq. (4.10).

3. It is clear that

$$\beta * u(x) = \int_0^\infty (\beta(x+y) - \beta(y))u(y)\,\mathrm{d}y + \int_0^\infty (\beta(y) - \beta(y-x))u(y)\,\mathrm{d}y. \tag{4.15}$$

Eq. (3.3) implies

$$|\beta * u(x)| \leqslant c_1 x(1+x^r)\left(1 + \int_0^\infty y^r u(y)\,\mathrm{d}y\right), \quad x \geqslant 0. \tag{4.16}$$

Since $r+1 \leqslant 2q$ and $2q \leqslant p-2$, by Hölder inequality, we have

$$\gamma_r(u) = \int_{\mathbb{R}} |y|^r u(y)\,\mathrm{d}y \leqslant (\gamma_{2q}(u))^{r/2q} \leqslant (MC_1)^{r/2q}$$
$$\leqslant (MC_1)^{2q-1)/2q} \leqslant M^{(1+1/q)(1-1/2q)} \leqslant M.$$

By integration, we easily verify Eq. (4.10). $\square$

**Remark.** In Eq. (4.10), it is important to have $x^2(1+x^{2q})$. An upper bound of the following type: $1 + x^{2q+2}$ is not sufficient (see, for instance, the proof of Lemma 4, and especially Eq. (4.17)).

**Lemma 4.5.** *There exists $M$ depending only on $\beta$ such that $\mathscr{A}(\mathscr{C}_M) \subset \mathscr{C}_M$.*

**Proof of Lemma 4.5.** We set $v(x) = (1 + |x|^p)\mathscr{A}u(x)$, $u$ belonging to $\mathscr{C}_M$. Using Eq. (4.10), we obtain

$$0 \leqslant v(x) \leqslant \frac{1}{\lambda(u)}(1 + |x|^p)\exp - \int_0^{|x|} \beta(y)\,\mathrm{d}y.$$

Therefore $\sup_{x \in \mathbb{R}} |v(x)| \leqslant c_3/\lambda(u)$, where

$$c_3 = \sup_{x \in \mathbb{R}} (1 + |x|^p)\exp - \int_0^{|x|} \beta(y)\,\mathrm{d}y.$$

Using Eq. (4.10) once more, and the definition of $\lambda(u)$,

$$\lambda(u) \geqslant \int_0^{+\infty} e^{-C_2 M x^2 (1+x^{2q})} dx.$$

We set $x\sqrt{M} = y$ in the integral

$$\lambda(u) \geqslant \frac{1}{\sqrt{M}} \int_0^{+\infty} e^{-C_2 y^2 (1+(y^2/M)^q)}\, dy \geqslant \frac{c_4}{\sqrt{M}}, \tag{4.17}$$

$$c_4 = \int_0^{+\infty} e^{-C_2 y^2 (1+y^{2q})}\, dy.$$

Finally, we get

$$\sup_{x\in\mathbb{R}} |v(x)| \leqslant \frac{c_3}{c_4}\sqrt{M}.$$

If we choose $M \geqslant \max((c_3/c_4)^2, 1, C_1^q)$ then $\sup_x (1+|x|^p)\mathscr{A}u(x) \leqslant M$. Since $\mathscr{A}u$ is an even density function, the result is proved. □

**Lemma 4.6.** *$\mathscr{A}$ is a continuous operator.*

**Proof of Lemma 4.6.**
1. Let $u, v$ in $\mathscr{C}_M$. We introduce

$$\theta(x) = e^{-(\int_0^x (\beta * u)(y)\, dy)} - e^{-(\int_0^x (\beta * v)(y)\, dy)}, \quad x \in \mathbb{R}.$$

Let $x \geqslant 0$. Using Eq. (4.13), we have

$$\theta(x) = e^{(-\int_0^x \beta(y)\, dy)}[e^{-\varphi_u(x)} - e^{-\varphi_v(x)}],$$

$$\varphi_w(x) = \int_0^x \widetilde{\beta}_w(y)\, dy\,;\ \widetilde{\beta}_w(y) = \int_0^\infty (\beta(y+t) - \beta(t-y) - 2\beta(y))\, w(t)\, dt;$$
$w = u$ or $v$.

Since $|e^{-a} - e^{-b}| \leqslant |a-b|$; $a, b \geqslant 0$, and $\beta$ verifies Eq. (4.8),

$$|\theta(x)| \leqslant e^{(-\int_0^x \beta(y)\, dy)} \int_0^x H(y)\, dy,$$

where

$$\begin{aligned} H(y) &= \left|\int_0^\infty (\beta(y+t) - \beta(t-y) - 2\beta(y))(u(t) - v(t))\, dt\right|; \quad y \geqslant 0 \\ &= \left|\int_0^\infty (\beta(y+t) - \beta(t-y))(u(t) - v(t))\, dt\right|. \end{aligned}$$

Then

$$H(y) \leqslant \int_0^\infty |\beta(y+t) - \beta(t-y)||u(t) - v(t)|\, dt.$$

We have

$$|u(t)-v(t)|^{1/2} \leqslant \sqrt{2M}(1+t^p)^{-1/2}.$$

Since $|\beta(y+t)-\beta(t-y)| \leqslant cy(1+y^r)(1+t^r)$, and $p>4q$,

$$H(y) \leqslant c_1 y(1+y^r)\|u-v\|_\infty^{1/2}.$$

After integration, we obtain

$$|\theta(x)| \leqslant c\mathrm{e}^{-(\int_0^x \beta(y)\,\mathrm{d}y)} x^2(1+x^r)\|u-v\|_\infty^{1/2}, \tag{4.18}$$

$$|\theta(x)| \leqslant c_3\|u-v\|_\infty^{1/2}. \tag{4.19}$$

2. We decompose $\mathscr{A}u(x)-\mathscr{A}v(x)$ as follows:

$$\mathscr{A}u(x)-\mathscr{A}v(x)=\frac{1}{\lambda(u)}\theta(x)+(\lambda(v)-\lambda(u))W(x), \tag{4.20}$$

$$W(x)=\frac{1}{\lambda(u)\lambda(v)}\exp-\int_0^x(\beta * v)(y)\,\mathrm{d}y.$$

By Eqs. (4.10), and (4.17),

$$0 \leqslant W(x) \leqslant \frac{M}{c_4^2}, \quad x \geqslant 0.$$

Since $\lambda(u)-\lambda(v)=\int_0^\infty \theta(x)\,\mathrm{d}x$, using Eq. (4.18), it is now easy to check that $\mathscr{A}$ is a continuous operator. □

**Proof of Theorem 4.1.**

1. We claim that $\overline{\mathscr{A}(\mathscr{C}_M)}$ is compact. Using the definition of $\mathscr{A}u$, the derivative of this function is given by

$$(\mathscr{A}u)'(x)=-\frac{1}{\lambda(u)}(\beta * u)(x)\mathrm{e}^{-\int_0^x \beta * u(y)\,\mathrm{d}y}.$$

By Eqs. (4.10), (4.16), (4.17) and (3.4), there exists $c_6>0$ such that

$$|(\mathscr{A}u)'(x)| \leqslant c_5(1+|x|^{2q+1})\mathrm{e}^{-c_6x^2}. \tag{4.21}$$

Let $(u_n)_{n\geqslant 1}$ be a sequence of functions belonging to $\mathscr{C}_M$. By the Ascoli theorem, there exists a sub-sequence (for simplicity, we denoted it $(u_n)_{n\geqslant 1}$) such that $\mathscr{A}u_n$ converges to $v$. By Eq. (4.21), it is easy to check that $\mathscr{A}u_n$ converges to $v$ in $\mathscr{B}$. This shows that $\overline{\mathscr{A}(\mathscr{C}_M)}$ is compact.

2. We are allowed to apply Proposition 4.2: There exists $\mu \in \mathscr{C}_M$ such that $\mathscr{A}\mu=\mu$. That shows point 1. of Theorem 4.1. Obviously, $\mu$ is $C^1$, and

$$\mu'(x)=-(\beta * \mu)(x)\mu(x).$$

As a result, $\mu$ satisfies Eq. (4.1). Suppose that $P(X_0\in \mathrm{d}x)=\mu(x)\,\mathrm{d}x$. Then $P(X_t\in \mathrm{d}x)=\mu(x)\,\mathrm{d}x$, since

$$b(t,x)=E[\beta(x-X_t)]=E[\beta(x-X_0)]=(\beta * \mu)(x).$$

We suppose that $\beta : \mathbb{R}_+ \to \mathbb{R}_+$ is a convex function. Hence $x \to \beta(x)/x$ is an increasing function on $]0,\infty[$, and $\alpha = \lim_{x\to 0_+} \beta(x)/x$ exists and belongs to $[0,\infty[$. We set

$$\beta(x) = \beta_0(x) + \alpha x. \tag{4.22}$$

Obviously, $\beta_0$ is convex, and $\lim_{x\to 0_+} \beta_0(x)/x = 0$. We now investigate uniqueness in Eq. (4.3). We suppose that $\beta$ is convex on $\mathbb{R}_+$. Let Eq. (4.22) be the decomposition of $\beta$, where $\alpha > 0$, and $\beta_0$ is an odd and increasing function, verifying Eq. (3.3). Obviously, $\beta$ is also an odd increasing function, and Lipschitz-continuous.

**Theorem 4.7.** *Assume that $\beta$ admits the decomposition* (4.22) *and* $\lim_{x\to 0_+} \beta_0(x)/x = 0$. *There then exists $\alpha_{\beta_0} > 0$ such that, for any $\alpha \geqslant \alpha_{\beta_0}$, Eq.* (4.3) *admits at most one solution.*

**Remark 4.8.** If $\beta$ is moreover a convex function on $\mathbb{R}_+$ (which is equivalent to $\beta_0$ is a convex function on $\mathbb{R}_+$), then Eq. (4.3) admits a unique solution.

To prove Theorem 4.8, a functional subspace of $\mathscr{D}$ (the definition of $\mathscr{D}$ is given in Lemma 4.9 below) are defined. The restriction of $\mathscr{A}$ to this subspace is a contraction operator. These results are stated in Lemmas 4.9–4.12.

Eq. (4.22) allows us to obtain a new lower bound for $\int_0^x \beta * u(y)\,\mathrm{d}y$. More precisely:

**Lemma 4.9.** *Assume that* Eq. (4.22) *holds and $u \in \mathscr{D}$, where*

$$\mathscr{D} = \left\{ v : \mathbb{R} \to \mathbb{R}_+;\ \int_{\mathbb{R}} v(x)\,\mathrm{d}x = 1, v(x) = v(-x) \forall x \in \mathbb{R},\ \sup_x (1+|x|^{2n}) v(x) < \infty \right\},$$

*then*

(i) $\beta_0 * u(x) = \int_0^\infty (\beta_0(x+y) - \beta_0(y-x)) u(y)\,\mathrm{d}x \geqslant 0, \quad \forall x \geqslant 0,$
(ii) $\beta * u(x) = \beta_0 * u(x) + \alpha x; \quad \beta * u(x) \geqslant \alpha x \quad \forall x \geqslant 0.$

**Proof of Lemma 4.9.** Since $\beta_0$ (resp. $u$) is an odd (resp. even) function, (i) follows immediately. As $\beta_0$ is increasing, then $\beta_0 * u(x) \geqslant 0$ if $x \geqslant 0$. $\square$

Let $u$ be a solution of Eq. (4.3). By Lemma 4.9,

$$u(x) \leqslant \frac{1}{\lambda(u)} \mathrm{e}^{-\alpha x^2/2}, \quad x \geqslant 0. \tag{4.23}$$

Therefore $u$ belongs to $\mathscr{D}_\alpha(\mathscr{A})$, where

$$\mathscr{D}_\alpha(\mathscr{A}) = \left\{ u : \mathbb{R} \to \mathbb{R}_+, \int_{\mathbb{R}} u(x)\,\mathrm{d}x = 1, u(-x) = u(x) \quad \forall x \in \mathbb{R}, u \text{ verifying (4.23)} \right\}. \tag{4.24}$$

We equip $\mathscr{D}_\alpha(\mathscr{A})$ with the norm

$$N_p(u) = \int_0^\infty x(1+x^p)|u(x)|\,\mathrm{d}x, \quad p > 4q. \tag{4.25}$$

Since $\mathscr{A}$ is an operator from $\mathscr{D}_\alpha(\mathscr{A})$ to $\mathscr{D}$, the idea of the proof is to show that if $\alpha$ is large enough, $\mathscr{A}$ is a contraction. As in the proof of Theorem 4.1, we need an upper bound to $1/\lambda(u)$.

**Lemma 4.10.** *There exists a positive constant $c_{\beta_0}$ such that, for any $u$ in $\mathscr{D}_\alpha(\mathscr{A})$:*

$$\frac{1}{\lambda(u)} \leqslant \alpha c_{\beta_0} \quad \forall \alpha \geqslant 1.$$

**Proof of Lemma 4.10.** Replacing $\beta$ by $\beta_0$ in Eq. (4.16), we obtain

$$\beta_0 * u(y) \leqslant cy(1+y^r)\left(1+\int_0^\infty t^r u(t)\,\mathrm{d}t\right), \quad y \geqslant 0,$$

where $c$ is a positive constant.

Since $u$ verifies Eq. (4.23), after integration we obtain

$$\int_0^x \beta_0 * u(y)\,\mathrm{d}y \leqslant c' x^2(1+x^r)\left(1+\frac{1}{\lambda(u)}\right), \quad x \geqslant 0. \tag{4.26}$$

Lemma 4.9 implies

$$\lambda(u) \geqslant 2\int_0^\infty \exp -\left(c'x^2(1+x^r)\left(1+\frac{1}{\lambda(u)}\right)+\alpha\frac{x^2}{2}\right)\mathrm{d}x.$$

We set $\mu = \sqrt{\lambda(u)}$, and $x = \mu y$ in the integral and easily see that

$$\begin{aligned} &\mu \geqslant h(\mu),\\ &h(t) = 2\int_0^\infty \exp -\left\{c' y^2(1+(ty)^r)(1+t^2)+\frac{\alpha t^2}{2}y^2\right\}\mathrm{d}y. \end{aligned} \tag{4.27}$$

Evidently, $h$ is a decreasing function on $[0,\infty[$; moreover,

$$\begin{aligned} -h'(t) \leqslant\ & 2\int_0^\infty \{c' y^2[r y^r t^{r-1}(1+t^2)+2t(1+(ty)^r)]+\alpha y^2 t\}\\ & \times \exp -y^2\left(c'+\frac{\alpha t^2}{2}\right)\mathrm{d}y. \end{aligned}$$

Assume that $t \in [0,1]$. Then

$$-h'(t) \leqslant c_1 + 2\alpha t\int_0^\infty y^2 \exp -y^2\left(c'+\frac{\alpha t^2}{2}\right)\mathrm{d}y,$$

with $c_1$ depending only on $r$, $c'$.

We set $x = y\sqrt{c'+\alpha t^2/2}$. Then

$$-h'(t) \leqslant c_1 + \frac{2\alpha t}{(c'+\alpha t^2/2)^{3/2}}\int_0^\infty x^2 \mathrm{e}^{-x^2}\,\mathrm{d}x \leqslant c_1 + c_2\alpha\rho_\alpha(t),$$

where

$$\rho_\alpha(t) = \frac{t}{c'+\alpha t^2/2}.$$

We observe that

$$\rho_\alpha(t) \leqslant \rho_\alpha\left(c\sqrt{\frac{2}{\alpha}}\right) = \frac{c_3}{\sqrt{\alpha}}.$$

Therefore,

$$h(0) - h(t) \leqslant t(c_1 + c_4\sqrt{\alpha}).$$

Since $h(0) = c_6$, $h(t) \geqslant c_7(1 - t(1+\sqrt{\alpha}))$. As a result, if $t < \inf\{1, c_7(c_7(1+\sqrt{\alpha})+1)^{-1}\}$, then $h(t) > t$. Eq. (4.27) implies that $\mu = \sqrt{\lambda(u)} \geqslant c_7(c_7(1+\sqrt{\alpha})+1)^{-1}$, if $\alpha$ is large. This ends the proof of Lemma 4.10. □

**Lemma 4.11.** *Let $\theta : \mathbb{R} \to \mathbb{R}$ be the even function defined by*

$$\theta(x) = \exp\left(-\int_0^x \beta * u(y)\,\mathrm{d}y\right) - \exp\left(-\int_0^x \beta * v(y)\,\mathrm{d}y\right), \quad x \geqslant 0,$$

*with $u$ and $v$ being two elements in $\mathscr{D}_\alpha(\mathscr{A})$. Then*

$$|\theta(x)| \leqslant c x^2 (1 + x^r) \mathrm{e}^{-\alpha x^2/2} N_p(u - v).$$

**Proof of Lemma 4.11.** Using Lemma 4.9,

$$\theta(x) = \mathrm{e}^{-\alpha x^2/2}\theta_0(x),$$

$\theta_0$ is defined as $\theta$, $\beta$ being replaced by $\beta_0$. Our approach is similar to those developed in the first part of the proof of Lemma 4.6:

$$|\theta_0(x)| \leqslant \int_0^x H_0(y)\,\mathrm{d}y, \quad x \geqslant 0,$$

$$H_0(y) \leqslant \int_0^\infty |\beta_0(y + t) - \beta_0(t - y)||u(t) - v(t)|\,\mathrm{d}t, \quad y \geqslant 0.$$

Since $t \to (1 + t^r)/(1 + t^p)$ is bounded, the Lipschitz continuous property of $\beta_0$ implies

$$H_0(y) \leqslant c y (1 + y^r) N_p(u - v).$$

The required result follows immediately. □

**Lemma 4.12.** (1) *$\mathscr{A}$ is an operator $\mathscr{D}_\alpha(\mathscr{A}) \to \mathscr{D}_p$. $\mathscr{D}_\alpha(\mathscr{A})$ (resp. $\mathscr{D}_p$) being defined by Eq.* (4.24) (*resp.* $\mathscr{D}_p = \{v : \mathbb{R} \to \mathbb{R}_+;\ \int_\mathbb{R} v(x)\,\mathrm{d}x = 1, v(x) = v(-x) \quad \forall x \in \mathbb{R}, \sup_x[(1 + |x|^{2+p})v(x)] < \infty\}$).

(2) *There exists $\alpha_{\beta_0} > 0$, $0 < k_{\beta_0} < 1$ such that*

$$N_p(\mathscr{A}u - \mathscr{A}v) \leqslant k_{\beta_0} N_p(u - v),$$

*for any $u$, $v$ in $\mathscr{D}_\alpha(\mathscr{A})$, where $N_p(w) = \int_\mathbb{R} |x|(1 + |x^p|)|w(x)|\mathrm{d}x$.*

**Proof of Lemma 4.12.** By Lemmas 4.10 and 4.11, we have

$$\frac{1}{\lambda(u)}N_p(\theta)\leqslant c\alpha N_p(u-v)I_1, \qquad |\lambda(u)-\lambda(v)|\leqslant cI_2N_p(u-v)$$

$$I_1=\int_0^\infty x^3(1+x^r)(1+x^p)\mathrm{e}^{-\alpha x^2/2}\mathrm{d}x, \quad I_2=\int_0^\infty x^2(1+x^r)\mathrm{e}^{-\alpha x^2/2}\mathrm{d}x.$$

We set $\sqrt{\alpha}x=y$ and choose $\alpha>1$. Then

$$\frac{1}{\lambda(u)}N_p(\theta)\leqslant \frac{c}{\alpha}N_p(u-v), \quad |\lambda(u)-\lambda(v)|\leqslant \frac{c}{\alpha^{3/2}}N_p(u-v).$$

In the same way,

$$N_p(W)\leqslant c'\alpha,$$

$W$ being defined by Eq. (4.20). Therefore Eq. (4.20) implies

$$N_p(\mathscr{A}u-\mathscr{A}v)\leqslant \frac{c''}{\sqrt{\alpha}}N_p(u-v).$$

If $\alpha$ is large enough, $\mathscr{A}$ is a contraction.

The fixed point, if it exists, is unique. This ends the proof of Theorem 4.7. $\square$

Before ending this section, we would like to examine two cases: $\beta(x)=x^3$ and $\beta(x)=x^5$. Obviously, Theorem 4.1 can be applied as there exists invariant probability. However, Eq. (4.21) is not satisfied, since we do not know, in theory that the invariant probability is unique. We prove existence and uniqueness directly, and the proof in these two specific cases is very different from the general one. We could also analyze $\beta(x)=x^7$, but the proof is tedious.

**Proposition 4.13.** *Assume that $\beta(x)=x^3$ and $X$ is the Markov process solution of Eq.* (E). *Then $X$ admits a unique invariant probability $\mu(x)\,\mathrm{d}x$, $\mu$ being symmetric. Moreover,*

$$\mu(x)=\frac{\exp-(\frac{x^4}{4}+\frac{3x^2m_2}{2})}{\int_{\mathbb{R}}\exp-(\frac{x^4}{4}+\frac{3x^2m_2}{2})\,\mathrm{d}x},$$

*where $m_2$ is the unique positive solution of*

$$\int_{\mathbb{R}}x^2\exp-\left(\frac{x^4}{4}+\frac{3x^2m_2}{2}\right)\,\mathrm{d}x=m_2\int_{\mathbb{R}}\exp-\left(\frac{x^4}{4}+\frac{3x^2m_2}{2}\right)\,\mathrm{d}x.$$

**Proof of Proposition 4.13.** $\mu$ being an even function,

$$\beta*\mu(y)=\int_{\mathbb{R}}(y-x)^3\mu(x)\,\mathrm{d}x=y^3+3ym_2,$$

where

$$m_2=\int_{\mathbb{R}}y^2\mu(y)\,\mathrm{d}y. \tag{4.28}$$

If $\mu(y)\,\mathrm{d}y$ is an invariant measure, it solves Eq. (4.3), i.e.

$$\mu(x)=\frac{\exp-(\frac{x^4}{4}+\frac{3x^2m_2}{2})}{\int_{\mathbb{R}}\exp-(\frac{x^4}{4}+\frac{3x^2m_2}{2})\,\mathrm{d}x}. \tag{4.29}$$

Therefore there exists a unique symmetric probability $\mu(y)\,\mathrm{d}y$ if and only if there exists a unique $m_2>0$ verifying both Eqs. (4.28) and (4.29). We introduce

$$\psi(m)=\frac{\int_{\mathbb{R}}x^2\mathrm{e}^{-(\frac{x^4}{4}+3\frac{m}{2}x^2)}\,\mathrm{d}x}{\int_{\mathbb{R}}\mathrm{e}^{-(\frac{x^4}{4}+3\frac{m}{2}x^2)}\mathrm{d}x},\quad m>0.$$

The derivative of $\psi$ is given by

$$\psi'(m)=-\frac{3}{2}\left[\left(\int_{\mathbb{R}}x^4\nu(\mathrm{d}x)\right)-\left(\int_{\mathbb{R}}x^2\nu(\mathrm{d}x)\right)^2\right],$$

with

$$\nu(\mathrm{d}x)=\frac{1}{c}\exp-\left(\frac{x^4}{4}+\frac{3m}{2}x^2\right)\,\mathrm{d}x,$$

$$c=\int_{\mathbb{R}}\exp-\left(\frac{x^4}{4}+\frac{3m}{2}x^2\right)\,\mathrm{d}x.$$

The Schwarz inequality tells us $\psi'(m)<0$. $\psi$ is a decreasing function, and $\psi(0)>0$. Therefore, there exists a unique $m$ such that $\psi(m)=m$. That ends the proof of Proposition 4.13. □

**Proposition 4.14.** *Assume that $\beta(x)=x^5$, and $X$ is the solution of Eq.* (E). *Then $X$ admits a unique invariant and symmetric probability measure $\mu(x)\,\mathrm{d}x$, given by*

$$\mu(x)=\frac{\exp-(\frac{x^6}{6}+\frac{5}{2}m_2x^4+\frac{5}{2}m_4x^2)}{\int_{\mathbb{R}}\exp-(\frac{x^6}{6}+\frac{5}{2}m_2x^4+\frac{5}{2}m_4x^2)\,\mathrm{d}x}, \tag{4.30}$$

*where $(m_2,m_4)$ is the unique positive solution of*

$$m_2\int_{\mathbb{R}}\exp-\left(\frac{x^6}{6}+\frac{5}{2}m_2x^4+\frac{5}{2}m_4x^2\right)\mathrm{d}x$$
$$=\int_{\mathbb{R}}x^2\exp-\left(\frac{x^6}{6}+\frac{5}{2}m_2x^4+\frac{5}{2}m_4x^2\right)\mathrm{d}x \tag{4.31}$$

$$m_4\int_{\mathbb{R}}\exp-\left(\frac{x^6}{6}+\frac{5}{2}m_2x^4+\frac{5}{2}m_4x^2\right)\mathrm{d}x$$
$$=\int_{\mathbb{R}}x^4\exp-\left(\frac{x^6}{2}+\frac{5}{2}m_2x^4+\frac{5}{2}m_4x^2\right)\mathrm{d}x. \tag{4.32}$$

**Proof of Proposition 4.14.**

1. Since $\beta(x)=x^5$ and $\mu$ is an even function

   $\beta*\mu(x)=x^5+10x^3m_2+5xm_4,$

where

$$m_k = \int_{\mathbb{R}} y^k \mu(y)\,\mathrm{d}y.$$

By Eq. (4.3), $\mu$ is uniquely given by Eq. (4.30), $(m_2, m_4)$ verifying Eqs. (4.31) and (4.32).

We have to check that $(m_2, m_4)$ is unique. Let us introduce

$$\mu_k(m_2, m_4) = \int_{\mathbb{R}} x^k \exp - \left( \frac{x^6}{6} + \frac{5}{2} m_2 x^4 + \frac{5}{2} m_4 x^2 \right), \tag{4.33}$$

$$\hat{\mu}_k = \frac{\mu_k}{\mu_0}. \tag{4.34}$$

Taking the derivative and using Hölder inequalities, we have

$$\frac{\partial \hat{\mu}_{2k}}{\partial m_2} = -\frac{5}{2}(\hat{\mu}_{2k+4} - \hat{\mu}_{2k}\hat{\mu}_4) < 0; \qquad \frac{\partial \hat{\mu}_{2k}}{\partial m_4} = -\frac{5}{2}(\hat{\mu}_{2k+2} - \hat{\mu}_{2k}\hat{\mu}_2) < 0. \tag{4.35}$$

Starting with $\mu_0$ (resp. $\mu_2$), an integration by parts easily furnishes the following relations.

$$\hat{\mu}_6 = 1 - 5 m_4 \hat{\mu}_2 - 10 m_2 \hat{\mu}_4, \tag{4.36}$$

$$3\hat{\mu}_2 = 5 m_4 \hat{\mu}_4 + 10 m_2 \hat{\mu}_6 + \hat{\mu}_8. \tag{4.37}$$

2. Let $m_2 > 0$ be fixed. Since $\hat{\mu}_2(m_2, .)$ is decreasing, there exists a unique $m_4 = \varphi(m_2)$ such that

$$\hat{\mu}_2(m_2, \varphi(m_2)) = m_2. \tag{4.38}$$

if $\hat{\mu}_2(m_2, 0) > m_2$. As $\hat{\mu}_2(., 0)$ is a decreasing function, if $m_2 < \alpha$, then $\hat{\mu}_2(m_2, 0) > m_2$. As a result, Eq. (4.38) holds if $m_2$ is small enough.
In the same way, we easily prove the existence of a unique $m_4 = \psi(m_2)$ such that

$$\hat{\mu}_4(m_2, \psi(m_2)) = \psi(m_2). \tag{4.39}$$

If we differentiate Eqs. (4.38) and (4.39), we have

$$\varphi'(m_2) = (1 + \Delta)\left( \frac{\partial \hat{\mu}_2}{\partial m_2} \right)^{-1} < 0, \qquad \psi'(m_2) = \frac{1}{1 + \Delta} \frac{\partial \hat{\mu}_4}{\partial m_2}, \tag{4.40}$$

where

$$\Delta = \frac{5}{2}(\hat{\mu}_6 - \hat{\mu}_2\hat{\mu}_4) = -\frac{\partial \hat{\mu}_2}{\partial m_2} = -\frac{\partial \hat{\mu}_4}{\partial m_4} > 0. \tag{4.41}$$

3. At this stage, we have to show that there exists a unique $m_2 > 0$ verifying

$$\psi(m_2) = \varphi(m_2). \tag{4.42}$$

Let $m_2 > 0$, verifying $\psi(m_2) = \varphi(m_2)$. We claim that

$$-\varphi'(m_2) > -\psi'(m_2). \tag{4.43}$$

By Eqs. (4.40), (4.43) is equivalent to

$$(1+\Delta)^2 \geqslant \left(\frac{\partial\hat{\mu}_2}{\partial m_2}\right)\left(\frac{\partial\hat{\mu}_4}{\partial m_2}\right).$$

We take the derivative of Eq. (5.36) with respect to $m_2$ and $m_4$. Recalling that $\partial\hat{\mu}_6/\partial m_2 \leqslant 0$, $\partial\hat{\mu}_6/\partial m_4 \leqslant 0$, we have

$$\frac{-\partial\hat{\mu}_4}{\partial m_2} \leqslant -\frac{m_4}{2m_2}\Delta + \frac{1}{m_2}\hat{\mu}_4,$$

$$\frac{-\partial\hat{\mu}_2}{\partial m_4} \leqslant \frac{1}{m_4}\hat{\mu}_2 - \frac{2m_2}{m_4}\Delta.$$

Therefore,

$$\left(\frac{-\partial\hat{\mu}_2}{\partial m_2}\right)\left(\frac{\partial\hat{\mu}_4}{\partial m_2}\right) \leqslant \left(1-\frac{\Delta}{2}\right)(1-2\Delta) \leqslant (1+\Delta)^2,$$

since $\hat{\mu}_4 = m_4$ and $\hat{\mu}_2 = m_2$.

4. The last step of the proof consists in proving that if $f : \mathbb{R}_+ \to \mathbb{R}_+$, $g : \mathbb{R}_+ \to \mathbb{R}_+$ are two decreasing functions of class $C^1$ verifying $\{t \geqslant 0; f(t) = g(t)\} \subset \{t \geqslant 0, -f'(t) > -g'(t)\}$, then there exists at most one $t$ such that $f(t) = g(t)$. We set $h(x) = f(x) - g(x)$. If $h(x) = 0$, then $h'(x) > 0$ and $h(y) > 0$ belonging to $]x, x+\varepsilon]$, for some $\varepsilon > 0$. Let $t_1 < t_2$, $h(t_1) = h(t_2) = 0$. We define $s := \inf\{u \in ]t_1, t_2]; h(u) = 0\}$. Obviously, $h(t_1) = h(s) = 0$ and $h(u) > h(s)$, for any $u \in [t_1, s[$. Consequently $h'(s) \leqslant 0$, which generates a contradiction. □

## 5. System of particles associated with Eq. (E)

For every integer $N \geqslant 1$, we consider the following $N$-dimensional SDE

$$\begin{cases} X_t^{i,N} = X_0^i + B_t^i - \dfrac{1}{2}\displaystyle\int_0^t \frac{1}{N}\left(\sum_{j=1}^N \beta(X_s^{i,N} - X_s^{j,N})\right) \mathrm{d}s, \quad t \geqslant 0 \\ 1 \leqslant i \leqslant N \end{cases} \qquad (\mathrm{S}_{\mathrm{N},\beta})$$

where $B = (B^1, \ldots, B^N)$ is a $\mathbb{R}^N$-valued standard Brownian motion. We assume that $X_0^1, \ldots, X_0^N$ are independent and have the same distribution. $\beta$ is as in Section 2, verifying Eqs. (3.1)–(3.4).

If $\beta$ is a bounded and locally Lipschitz continuous function, the system $(\mathrm{S}_{\mathrm{N},\beta})$ has a unique strong solution for each $N$; the propagation of chaos holds; and the limit law of $X^{1,N}$ is the law of the unique strong solution of a nonlinear SDE (Sznitman, 1989). Unfortunately, $\beta$ is not bounded. The first difficulty is equivalent of showing that Eq. $(\mathrm{S}_{\mathrm{N},\beta})$ admits a strong solution.

**Proposition 5.1.** (1) *Eq.* $(\mathrm{S}_{\mathrm{N},\beta})$ *has a unique strong solution.*

(2) *If we assume that* $E(|X_0|^p)<\infty$ *for every* $p\geqslant 1$, *then*

$$\sup_{0\leqslant t\leqslant T} E\left[\left(\sum_{i=1}^{N}(X^{i,N}(t))^2\right)^p\right]\leqslant c(p,T). \tag{5.1}$$

**Proof of Proposition 5.1.** (a) *We define*

$$\beta_n(x)=\begin{cases}\beta(x) & \text{if } |x|\leqslant n,\\ \beta(n) & \text{if } x>n,\\ \beta(-n) & \text{if } x<n.\end{cases}$$

The drift term in $(\mathrm{S}_{\mathrm{N},\beta_n})$ is actually bounded, and is a Lipschitz function. Therefore, Eq. $(\mathrm{S}_{\mathrm{N},\beta_n})$ has a unique strong solution $X_n=(X_n^1,\dots,X_n^N)$. Let us denote its first exit time of $\{x;\|x\|\leqslant n\}$ by $T_n$:

$$T_n=\inf\{t\geqslant 0;|X_n(t)|>n\}.$$

As in the proof of Proposition 3.3, we have to show that $\sup T_n=\infty$. We apply the Itô formula to $\sum_{i=1}^N X_n^i(t)^2$:

$$\sum_{i=1}^N X_n^i(t)^2=\sum_{i=1}^N X_n^i(0)^2+2\sum_{i=0}^N\int_0^t X_n^i(s)\,\mathrm{dB}_s^i+Nt$$
$$-\frac{1}{N}\int_0^t\left(\sum_{i,j=1}^N X_n^i(s)\beta(X_n^i(s)-X_n^j(s))\right)\mathrm{d}s.$$

With $\beta$ satisfying Eqs. (3.1) and (3.2), we observe that

$$\sum_{i,j=1}^N x^i\beta(x^i-x^j)=\sum_{1\leqslant i<j\leqslant N}(x^i\beta(x^i-x^j)+x^j\beta(x^j-x^i))$$
$$=\sum_{1\leqslant i<j\leqslant N}(x^i-x^j)\beta(x^i-x^j)\geqslant 0.$$

Therefore,

$$|X_n^i(t)|^2\leqslant|X_n(0)|^2+2\sum_{i=0}^N\int_0^t X_n^i(s)\,\mathrm{dB}_s^i+Nt.$$

As in the proof of Proposition 3.3, the above inequality implies that $\sup T_n=\infty$. a.s.

(b) We now check Eq. (5.1). As $N$ is fixed, we set $X^i=X^{i,N}$, $1\leqslant i\leqslant N$ for simplicity. We apply the Itô formula to $\sum_{i=1}^N X^i(t)^{2p+2}$:

$$\sum_{i=1}^N X^i(t)^{2p+2}=\sum_{i=1}^N X^i(0)^{2p+2}-\frac{p+1}{N}\int_0^t\left(\sum_{1\leqslant i,j\leqslant N}X^i(s)^{2p+1}\beta(X_s^i-X_s^j)\right)\mathrm{d}s$$
$$+(p+1)(2p+1)\int_0^t\left(\sum_{i=1}^N X^i(s)^{2p}\right)\mathrm{d}s+M_t,$$

where $M$ is a continuous local martingale.

As in step (a),

$$\sum_{1\leqslant i,j\leqslant N} (x^i)^{2p+1}\beta(x^i-x^j)=\sum_{1\leqslant i<j\leqslant N} ((x^i)^{2p+1}-(x^j)^{2p+1})\beta(x^i-x^j)\geqslant 0.$$

Therefore, if we set $\varphi_p(t)=E[\sum_{i=1}^n X^i(t)^{2p}]$, $\varphi_p$ satisfies

$$\varphi_{p+1}(t)\leqslant\varphi_{p+1}(0)+(p+1)(2p+1)\int_0^t \varphi_p(s)\,\mathrm{d}s. \tag{5.2}$$

Since $\varphi_0(t)=n$, it is easy to check by induction on $p$ that $c_p(T)=\sup_{0\leqslant t\leqslant T}\varphi_p(t)<\infty$. □

**Remark 5.2.** We observe that we do not use the fact that $\beta$ verifies Eq. (3.4) in the proof of Proposition 5.1.

Let $\overline{X}^i$ be the solution of Eq. (E) with initial data $X_0^i$:

$$\begin{aligned}&\overline{X}_t^i=X_0^i+B_t^i-\frac{1}{2}\int_0^t b(s,\overline{X}_s^i)\,\mathrm{d}s,\\ &b(s,x)=E[\beta(x-\overline{X}_s^i)].\end{aligned} \tag{5.3}$$

We now state the main result of this section:

**Theorem 5.3.** *Assume that* $E(|X_0|^{2(r+1)^2})<\infty$. *Then there exists* $C(T)>0$, *such that*

$$E\left[\sup_{0\leqslant s\leqslant T}|X_s^{i,N}-\overline{X}_s^i|^2\right]\leqslant\frac{c(T)}{N}. \tag{5.4}$$

(*We recall that* $r$ *is a constant associated with* $\beta$ *and is defined by Eq.* (3.3).)

Theorem 5.3 will be proved in two steps.

**Lemma 5.4.** *There exists a constant* $C>0$ *such that for every* $N\geqslant 1$

$$\sup_{0\leqslant s\leqslant T} E(|X_s^{i,N}-\overline{X}_s^i|^2)\leqslant\frac{CT^2}{N}, \tag{5.5}$$

$$\sup_{0\leqslant s\leqslant T} E(|X_s^{i,N}-\overline{X}_s^i|^4)\leqslant\frac{CT^4}{N^2}. \tag{5.6}$$

**Proof of Lemma 5.4.** Since $X^{i,N}$ (resp. $\overline{X}^i$) is a solution of $\mathrm{S}_{\mathrm{N},\beta}$ (resp. Theorem 5.3),

$$X_t^{i,N}-\overline{X}_t^i=-\frac{1}{2N}\int_0^t\left[\sum_{j=1}^N\{\beta(X_s^{i,N}-X_s^{j,N})-b(s,\overline{X}_s^i)\}\right]\mathrm{d}s. \tag{5.7}$$

(1) Using the Itô formula, we obtain

$$\sum_{i=1}^N (X_t^{i,N}-\overline{X}_t^i)^2=-\frac{1}{N}\sum_{1\leqslant i,j\leqslant N}\int_0^t(\rho_{i,j}^{(1)}(s)+\rho_{i,j}^{(2)}(s))\,\mathrm{d}s, \tag{5.8}$$

where

$$\rho^{(1)}_{i,j}(s)=[\beta(X^{i,N}_s-X^{j,N}_s)-\beta(\overline{X}^i_s-\overline{X}^j_s)](X^{i,N}_s-\overline{X}^i_s),$$

$$\rho^{(2)}_{i,j}(s)=[\beta(\overline{X}^i_s-\overline{X}^j_s)-b(s,\overline{X}^i_s)](X^{i,N}_s-\overline{X}^i_s).$$

Analyzing the first sum involving $\rho^{(1)}_{i,j}$, we have

$$\sum_{1\leqslant i,j\leqslant N}\rho^{(1)}_{i,j}(s)=\sum_{1\leqslant i<j\leqslant N}\rho^{(3)}_{i,j}(s), \tag{5.9}$$

$$\rho^{(3)}_{i,j}(s)=\rho^{(1)}_{i,j}(s)+\rho^{(1)}_{j,i}(s).$$

Since $\beta$ is an odd function,

$$\rho^{(3)}_{i,j}(s)=(\beta(X^{i,N}_s-X^{j,N}_s)-\beta(\overline{X}^i_s-\overline{X}^j_s))(X^{i,N}_s-\overline{X}^i_s-(X^{j,N}_s-\overline{X}^j_s)). \tag{5.10}$$

If $x-y\geqslant x'-y'$ (resp. $x-y\leqslant x'-y'$), then $x-x'\geqslant y-y'$ (resp. $x-x'\leqslant y-y'$) and

$$[(x-y)-(x'-y')][\beta(x-x')-\beta(y-y')]\geqslant 0. \tag{5.11}$$

Consequently, $\rho^{(3)}_{i,j}(s)\geqslant 0$, and

$$\sum_{1\leqslant i,j\leqslant N}\rho^{(1)}_{i,j}\geqslant 0. \tag{5.12}$$

On the other hand, using the Schwarz inequality, we obtain

$$-E\left(\sum_{j=1}^{N}\rho^{(2)}_{i,j}(s)\right)\leqslant\{E((X^{i,N}_s-\overline{X}^i_s)^2)\theta_i(s)\}^{1/2}, \tag{5.13}$$

$$\theta_i(s)=E\left(\left\{\sum_{j=1}^{N}[\beta(\overline{X}^i_s-\overline{X}^j_s)-b(s,\overline{X}^i_s)]\right\}^2\right).$$

Developing $\theta_i(s)$, we get

$$\theta_i(s)=\sum_{j=1}^{N}\xi_{j,j}(s)+2\sum_{1\leqslant j<k\leqslant N}\xi_{j,k}(s),$$

$$\xi_{j,k}(s)=E((\beta(\overline{X}^i_s-\overline{X}^j_s)-b(s,\overline{X}^i_s))(\beta(\overline{X}^i_s-\overline{X}^k_s)-b(s,\overline{X}^i_s))).$$

If $j\neq k$, $\overline{X}^i$, $\overline{X}^j$, $\overline{X}^k$ are three independent copies of $\overline{X}^1$, recall that $b(s,x)=E(\beta(x-\overline{X}^1_s))$ therefore

$$\xi_{j,k}(s)=0\quad\text{if } j\neq k.$$

Since $\beta$ satisfies Eq. (3.8), Proposition 3.10 tells us that $\xi_{j,j}$ is bounded. Hence

$$\theta_i(s)\leqslant c_1N, \tag{5.14}$$

where $c_1$ is a constant depending only on $\beta$.

By symmetry, it is clear that

$$E((X_s^{i,N}-\overline{X}_s^i)^2)=E((X_s^{1,N}-\overline{X}_s^1)^2). \tag{5.15}$$

As a result, if we take the expectation in Eq. (5.8) by Eqs. (5.9), (5.12), (5.13) and (5.14), then

$$N\left(\sup_{0\leqslant t\leqslant T}E((X_t^{1,N}-\overline{X}_t^1)^2)\right)\leqslant\sqrt{NC_1}T\left\{\sup_{0\leqslant t\leqslant T}E((X_t^{1,N}-\overline{X}_t^1)^2)\right\}^{1/2},$$

and Eq. (5.5) follows immediately.

The proof of Eq. (5.6) is very similar to the above. Briefly, the changes are

$$\sum_{i=1}^N(X_t^{i,N}-\overline{X}_t^i)^4=-\frac{2}{N}\sum_{1\leqslant i\leqslant j\leqslant N}\int_0^t(\rho_{i,j}^{(4)}(s)+\rho_{i,j}^{(5)}(s))\,\mathrm{d}s$$

$$\rho_{i,j}^{(4)}(s)=\rho_{i,j}^{(1)}(s)(X_s^{i,N}-\overline{X}_s^i)^2,$$

$$\rho_{i,j}^{(5)}(s)=\rho_{i,j}^{(2)}(s)(X_s^{i,N}-\overline{X}_s^i)^2.$$

Analogous to Eqs. (5.12) and (5.13), we have

$$\sum_{1\leqslant i\leqslant j\leqslant N}\rho_{i,j}^{(4)}\geqslant 0$$

and

$$-E\left(\sum_{j=1}^N\rho_{i,j}^{(5)}(s)\right)\leqslant\{E[(X_s^{i,N}-\overline{X}_s^i)^4]\}^{3/4}\hat{\theta}_i(s)^{1/4},$$

respectively, where

$$\hat{\theta}_i(s)=E\left(\left[\sum_{j=1}^N\{\beta(\overline{X}_s^i-\overline{X}_s^j)-b(s,\overline{X}_s^i)\}\right]^4\right).$$

As in step 1,

$$\hat{\theta}_i(s)\leqslant N^2C_2.$$

It is now easy to see that Eq. (5.6) holds. □

**Proof of Theorem 5.3.** By Eq. (5.7), we have

$$(X_t^{1,N}-\overline{X}_t^1)^2=-\frac{1}{N}\sum_{j=1}^N\int_0^t(\rho_{1,j}^{(1)}(s)+\rho_{1,j}^{(2)}(s))\,\mathrm{d}s.$$

Consequently,

$$\sup_{0\leqslant t\leqslant T}(X_t^{1,N}-\overline{X}_t^1)^2\leqslant\frac{1}{N}\sum_{j=1}^N\int_0^T(|\rho_{1,j}^{(1)}(s)|+|\rho_{1,j}^{(2)}(s)|)\,\mathrm{d}s. \tag{5.16}$$

We start with the sum involving $\rho^{(2)}_{1,j}$. By Eq. (5.13),

$$E\left(\sum_{j=1}^{N}|\rho^{(2)}_{1,j}(s)|\right)\leqslant\{E[(X^{i,N}_s-\overline{X}^i_s)^2]\theta_i(s)\}^{1/2}.$$

Using Eqs. (5.5) and (5.14), we obtain

$$E\left(\sum_{j=1}^{N}|\rho^{(2)}_{1,j}(s)|\right)\leqslant C_3T,\quad 0\leqslant s\leqslant T. \tag{5.17}$$

The Schwarz inequality implies:

$$E(|\rho^{(1)}_{1,j}(s)|)\leqslant\{E[(X^{1,N}_s-\overline{X}^1_s)^2]E[(\beta(X^{1,N}_s-X^{j,N}_s)-\beta(\overline{X}^1_s-\overline{X}^j_s))^2]\}^{1/2}. \tag{5.18}$$

Using Eq. (3.3), Propositions 5.1, 3.10, and the Schwarz inequality again, we have

$$E[(\beta(X^{1,N}_s-X^{j,N}_s)-\beta(\overline{X}^1_s-\overline{X}^j_s))^2]\leqslant c_4(T)\{E[(X^{1,N}_s-\overline{X}^1_s)^4]\}^{1/2} \tag{5.19}$$

Eq. (5.4) now follows easily from Eqs. (5.16)–(5.19). □

**Remark 5.5.** There is classical proof (Sznitman, 1989) that the chaos propagation is a consequence of Theorem 5.3. This means that, for every fixed $i$, the distribution of $(X^{1,N},X^{2,N},\dots,X^{i,N})$ converges to $(\overline{X}^1,\overline{X}^2,\dots,\overline{X}^i)$.

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